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Axioms for generalized Farrell–Tate cohomology

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Abstract

In Klein (Math. Ann, 319 (2001) 421–456) we defined a variant of Farrell–Tate cohomology for a topological group G and any naive G-spectrum E by taking the homotopy cofiber of a certain norm map $D_G \wedge_{hG} E \to E^{hG}$. In this paper, we show how to axiomatize this theory. We then interpret the norm map as the assembly map for the homotopy fixed point functor $E \mapsto E^{hG}$.

MSC: Primary 20J06, 55N91; secondary 55P91, 55P42

1. Introduction

The Tate cohomology groups $\hat{H}^*(G;M)$ were introduced in order to exploit the similarities between the cohomology and the homology of groups. At first, they were defined for finite groups G equipped with a $\mathbb{Z}[G]$-module M. Later, Farrell [6] extended the definition to discrete groups having finite virtual cohomological dimension.

Based on a construction appearing in [8], one can associate a *generalized Farrell–Tate spectrum* E^{tG} to any topological (or discrete) group G and any naive G-spectrum E. In this way, we obtain generalized Farrell–Tate groups $\hat{H}^*(G;E)$ by taking the homotopy groups of E^{tG}. These groups have many of the formal properties of the classical theory (the classical case occurs when G is discrete and E is an Eilenberg–MacLane spectrum).

It is the purpose of this paper to show how to characterize E^{tG} by a short list of axioms.

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1.1. The axioms

Let $\mathbf{Sp}_G$ denote the category of naive G-spectra, where G denotes the realization of a simplicial group. Suppose we are given a functor

$$E \mapsto E^{tG}$$

from G-spectra to spectra which satisfies the following four axioms:

Axiom 1 (*Homotopy invariance*)**.** The functor $E \mapsto E^{tG}$ preserves weak equivalences.

A *weak equivalence* of G-spectra is a morphism which induces an isomorphism on homotopy groups.

Axiom 2 (*Vanishing on finite objects*)**.** E^{tG} is weakly contractible whenever E is a finite G-spectrum.

A G-spectrum is said to be *finite* if it is built up from the zero object by attaching a finite number of free cells.

This axiom requires some explanation: if G is a finite, discrete group and M is a free $\mathbb{Z}[G]$-module, then the Tate groups $\hat{H}^*(G;M)$ vanish (this uses [2, Chapter VI, 5.3]). In particular, if M is *finitely generated* and free, the Tate groups vanish. We justify Axiom 2 by thinking of the finite G-spectra as spectrum level versions of the finitely generated free $\mathbb{Z}[G]$-modules.

The spectrum level analogues of the free modules are the *induced* spectra, i.e., the G-spectra of the form $W \wedge G_+$, where W is an unequivariant spectrum. The reader might justifiably ask why we haven't demanded the stronger requirement:

Axiom 2′ (*Vanishing on induced objects*)**.** E^{tG} is weakly contractible whenever E is an induced G-spectrum.

We have two reasons for not requiring Axiom 2′. The first is that Axioms 1–4 are equivalent to Axioms 1, 2′, 3 and 4 whenever G happens to be a compact Lie group (this uses [8, Corollary 10.2]). The second reason is that in the case of a general topological group G, I know of no example of a functor $E \mapsto E^{tG}$ which satisfies Axioms 1, 2′, 3 and 4 (it seems likely to me that a modification of Axiom 4 would be needed if Axiom 2′ is imposed).

Weiss and Williams [12] also construct a variant of Farrell–Tate cohomology for arbitrary topological groups. Their functor satisfies Axioms 1, 2′ and 3. However, it is not obvious to me (and is probably not the case) that their functor satisfies Axiom 4 for arbitrary groups G. However, in the case when the input G-spectrum E has finite skeleta, it is not difficult to see that their construction is equivalent to the one given in this paper.

Axiom 3 (*Relation to group cohomology*)**.** The functor E^{tG} comes equipped with a natural transformation $E^{hG} \to E^{tG}$, where E^{hG} denotes the homotopy fixed point spectrum of G acting on E.

Axiom 4 (*Homology axiom*)**.** Let E_{tG} be the homotopy fiber of $E^{hG} \to E^{tG}$. Then the functor $E \mapsto E_{tG}$ is a homology theory.

Being a *homology theory* means:
- the value at the zero object is (weakly) contractible,
- the functor preserves homotopy cocartesian squares, and
- the functor commutes with filtered (homotopy) colimits.

(The first two conditions mean that the functor is *excisive.*)

Thus the functor $E \to E^{tG}$ can be viewed as the “difference” between a homology theory and a cohomology theory. In particular, $E \mapsto E^{tG}$ is excisive.

Theorem A (Existence)**.** *There exists a functor $E \mapsto E^{tG}$ satisfying Axioms* 1–4 *above*.

(Uniqueness). *Given another homotopy functor $E \mapsto E^{wG}$ satisfying Axioms* 1–4, *there exists a chain of weak equivalences $E^{tG} \xrightarrow{\simeq} E^{wG}$ which transfers the natural map $E^{hG} \to E^{tG}$ onto the natural map $E^{hG} \to E^{wG}$.*

Remark 1.1. It will be shown in Section 5 that the homotopy cofiber of the *norm map*

$$D_G \wedge_{hG} E \to E^{hG}$$

(see [8]) satisfies Axioms 1–4. Here D_G denotes the *dualizing spectrum* of G, which was defined in [8] to be the homotopy fixed spectrum of G acting on the suspension spectrum of G_+.

The uniqueness part of Theorem A says that $E \mapsto E^{tG}$ is unique in the homotopy category of functors equipped with a natural transformation from the homotopy fixed point functor. In a sense which we would not bother to make precise, the category of all factorizations $E^{hG} \to E^{tG} \xrightarrow{\simeq} E^{wG}$ has contractible realization.

Assembly. Let C be a Quillen model category [9]. In particular, the notion of homotopy cocartesian square makes sense in C. Let **Sp** denote the category of spectra. Suppose that

$$F : C \to \mathbf{Sp}$$

is a functor which preserves weak equivalences.

Definition 1.2 (cf. Weiss and Williams [11])**.** A natural transformation

$$\alpha_F : F^{\%} \to F$$

of homotopy functors is said to be an *assembly map* if it possesses the following properties:

- $F^{\%}$ is a homology theory.
- The map α_F is a "universal" approximation (from the left) of F by a homology theory (in the homotopy category of functors): i.e., given any natural transformation $G \to F$ with G a homology theory, there is a factorization $G \to F^{\%} \to F$ up to formal inversion of (pointwise) weak equivalence of functors.

Note that any two choices of assembly map are isomorphic in the homotopy category of functors.

Classically, C was taken to be the category of topological spaces, and F was the L-theory functor. In fact, Weiss and Williams prove that any homotopy invariant F from spaces to spectra has an assembly map. In this paper, we take C to be the category of G-spectra, and F will be the homotopy fixed point functor $E \mapsto E^{hG}$.

Theorem B. *The norm map $D_G \wedge_{hG} E \to E^{hG}$ is the assembly map for the homotopy fixed point functor $E \mapsto E^{hG}$.*

We can now reformulate part of [8, Theorem D] in this language. Restated, we obtain a solution to "Borel Conjecture" for the homotopy fixed point functor:

Corollary C. *Let G be a topological group with $\pi_0(G)$ finitely presented. Then the following are equivalent*:

- *the classifying space BG is finitely dominated,*
- *the assembly map for the homotopy fixed point functor is a weak equivalence at every object E,*
- *the assembly map for the homotopy fixed point functor is a weak equivalence when $E = S^0$ is the sphere spectrum* (*with trivial G-action*).

Outline: Section 2 is primarily language. In Section 3 we prove the existence part of Theorem A. In Section 4 we introduce a construction which enables us later to reinterpret the cofiber of the norm map. Section 5 contains the proof of the uniqueness part of Theorem A. In Section 6 we prove Theorem B. Section 7 discusses a variant of the norm map which is related to Carlsson's transfer for infinite groups.

2. Conventions

All spaces are assumed to be compactly generated. The term "topological group" means a group object in the category of compactly generated spaces. We will only consider those topological groups which arise as the geometric realizations of simplicial groups. (This is not a serious restriction, as a topological group can be replaced by the realization of its singularization, and the constructions of [8] are insensitive to such manipulations.)

A (naive) G-spectrum E consists of based, left G-spaces E_n, and based G-maps $\Sigma E_n \to E_{n+1}$, where by convention G acts trivially on the suspension coordinate of the domain. For technical reasons, we typically assume that the underlying spaces of E_n are CW complexes. If necessary, this can always be arranged by applying singularization then realization degreewise to the spaces in a G-spectrum.

A *morphism* $E \to E'$ of G-spectra consists of based G-maps $E_n \to E'_n$ which are compatible with the structure maps. A morphism is a *weak equivalence* if it induces an isomorphism on homotopy groups. Weak equivalences are indicated by $\xrightarrow{\sim}$, and a chain of weak equivalences is indicated by $\simeq$. Schwede [10] has shown that the above notion of weak equivalence arises from a Quillen model category structure on the category of G-spectra. We now describe the fibrant and cofibrant objects in this model structure.

The fibrant objects in this instance are the Ω-spectra (i.e., those G-spectra E such that the adjoint $E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence of spaces for all n). If E is a G-spectrum, it has a functorial *fibrant approximation* $E \xrightarrow{\sim} E^{\mathrm{f}}$ given by $E^{\mathrm{f}}_n := \operatorname{hocolim}_j \Omega^j E_{n+j}$.

To describe the cofibrant objects, let k be an integer. Define S^{k-1}_G to be the object which in degree j is given by $S^{k-1+j} \wedge G_+$ if $k-1+j \geqslant 0$, and a point otherwise (here the structure maps are obvious). Let D^k_G denote the object given by taking the degreewise cone on S^{k-1}_G. Then D^k_G is the *free cell* of dimension k. Given a morphism $f : S^{k-1}_G \to Y$, we can form the object $Y \cup_f D^k_G$, which is the effect of attaching a k-cell. The *cofibrant* G-spectra are then the (retracts of) spectra built up from the zero object by attaching free cells. Every G-spectrum E has a functorial *cofibrant approximation* $E^{\mathrm{c}} \xrightarrow{\sim} E$. (The procedure for constructing the latter is essentially the same as the one given in [5] for topological spaces; we omit the details.)

The *finite* G-spectra are those objects built up from the zero object by a finite number of free cell attachments.

The *homotopy orbit spectrum* E_{hG} of G-acting on E is given by $EG_+ \wedge_G E$, i.e., the spectrum which in degree n is $EG_+ \wedge_G E_n =$ the orbits of G acting E_n made free. The *homotopy fixed point spectrum* is given by $F(EG_+, E^{\mathrm{f}})^{hG} =$ the spectrum which in degree n is the function space of based G-maps from EG_+ to $(E^{\mathrm{f}})_n$.

The category of G-spectra has *homotopy limits* and *colimits* (cf. [1]). To describe the nth space of the homotopy colimit of a diagram $\alpha \mapsto E^\alpha$ of G-spectra, one simply takes the homotopy colimit of the associated diagram of nth spaces $\alpha \mapsto (E^\alpha)^{\mathrm{c}}_n$ (where $(E^\alpha)^{\mathrm{c}}$ is the effect of making E^α cofibrant). Similar remarks apply to homotopy limits.

We shall use handicrafted smash products of equivariant spectra (we only require associativity up to homotopy). If E is an H-spectrum, and W is a G-spectrum, then $E \wedge W$ is a $(G \times H)$-spectrum.

We refer the reader to [8] for a more detailed discussion of the category of G-spectra.

3. Proof of Theorem A (existence)

Recall from [8] that we defined E^{tG} to be the homotopy cofiber of a certain *norm map*

$$D_G \wedge_{hG} E \to E^{hG}$$

in which

- D_G is the *dualizing spectrum* of G. Recall that the latter is defined to be $S^0[G]^{hG}$ where $S^0[G]$ denotes the suspension spectrum of G_+, homotopy fixed points are taken using the G-action defined by translation using the left action of G on G_+ and the action of G on D_G is defined by translation with respect to the right action of G on G_+.
- The domain of the norm map is the homotopy orbit spectrum of G acting diagonally on the smash product of $D_G \wedge E$.
- The codomain of the norm map is the homotopy fixed point spectrum of G acting on E.

For reasons of space, we will not review the construction of the norm map (in any case, we shall later reconstruct it in Sections 4 and 5). However, as remarked in [8], there is a straightforward way to think of the construction, provided one is willing to accept that the homotopy category of G-spectra has *internal* function objects. The norm map may then be defined as the *composition pairing*

$$\hom(S^0, S^0[G]) \wedge_{S^0[G]} \hom(S^0[G], E) \to \hom(S^0, E),$$

where hom is taken in the homotopy category of G-spectra. Here S^0 denotes the sphere spectrum (with trivial G-action). In the homotopy category, the function object $\hom(S^0, S^0[G])$ is isomorphic to D_G, $\hom(S^0[G], E)$ is isomorphic to E and $\hom(S^0, E)$ is isomorphic to E^{hG}.

We now establish the existence part of A. Axiom 1 is a consequence of the fact that taking homotopy cofibers is homotopy invariant. Axiom 2 follows from [8, Theorem D] (since we showed there that the norm map is a weak equivalence for finitely dominated G-spectra). Axiom 3 results from the fact that there is an evident map from $E^{\mathrm{h}G}$ into the homotopy cofiber of the norm map. Axiom 4 is a consequence of the fact that the homotopy fiber of the map $E^{hG} \to E^{tG}$ is identified with $D_G \wedge_{hG} E$ and the functor $E \mapsto D_G \wedge_{hG} E$ is clearly a homology theory.

This establishes the existence part of Theorem A. It remains for us to establish the uniqueness part. □

4. A construction

We discuss here to give an alternate description of the cofiber of the norm map. This will be used to establish the uniqueness part of Theorem A.

4.1. Motivation

Suppose it were possible to write any G-spectrum E up to weak equivalence as a homotopy colimit

$$E \simeq \operatorname*{hocolim}_{\alpha} E_\alpha,$$

in which E_α are finite G-spectra. Assuming this, we define $E^{(\alpha)}$ to be the homotopy cofiber of the map $E_\alpha \to E$, then we get a spectrum

$$E^{t'G} = \operatorname*{hocolim}_{\alpha} (E^{(\alpha)})^{hG}.$$

If one could make the assignment $E \to E^{t'G}$ functorial and homotopy invariant, then an argument given below shows that $E^{t'G}$ and E^{tG} are naturally weak equivalent.

As a first guess at what this system of finite G-spectra might be, take the indexing category of the homotopy colimit to have objects $\alpha : E_\alpha \to E$ where E_α is any finite G-spectrum and α is any map of G-spectra. A morphism is then a map $f : E_\alpha \to E_\beta$ such that $\beta \circ f = \alpha$. There is then an evident map from the homotopy colimit of the diagram defined by $\alpha \mapsto E_\alpha$ into E.

However, this construction presents several technical difficulties. For one thing if E fails to be an Ω-spectrum, then there might not be enough maps into it. Secondly, the indexing category is not filtered, so it is difficult in general to identify the homotopy type of the homotopy colimit. To avoid these difficulties, we perform a simplicial version of the forgoing, which has better properties.[1] Thereafter, we use singularization and realization to give us a well-behaved construction for topological spectra.

4.2. The simplicial version

Let $G_\bullet$ be a simplicial group. A (*simplicial*) *$G_\bullet$-spectrum* is a collection of based simplicial sets $\{E_n\}_{n \geqslant 0}$ with (based, left) $G_\bullet$-action, together with equivariant structure maps $\Sigma E_n \to E_{n+1}$. Note that the realization $|E|$ of E taken degreewise has the structure of a topological G-spectrum, where $G = |G_\bullet|$. Similarly, the singularization $\operatorname{Sing} W$ of a topological G-spectrum W is a simplicial $G_\bullet$- spectrum.

A map $E \to E'$ of $G_\bullet$-spectra consists of equivariant maps $E_n \to E'_n$ which are compatible with the structure maps. A map $E \to E'$ is a *weak equivalence* if $|E| \to |E'|$ is a weak equivalence. Schwede [10] has shown that this notion of weak equivalence arises from a Quillen model structure on the category of $G_\bullet$-spectra.

We will mostly need to know what the fibrant and cofibrant objects are in this model structure. The *fibrant objects* are those $G_\bullet$-spectra E which are degree-wise Kan simplicial sets (after forgetting actions) and moreover, the adjoints $E_n \to \Omega E_{n+1}$ to the structure maps are weak homotopy equivalences. A *cofibrant object* is the retract of an object which is built up from the zero object by attaching free cells (equivalently, a cofibrant object E is one in which E_n is free away from the basepoint, and moreover,

[1] I would like to thank S. Schwede for explaining to me why the simplicial version has better properties.

the structure map $\Sigma E_n \to E_{n+1}$ is a free map, i.e., E_{n+1} is obtained from E_n by attaching free $G_\bullet$-cells). An object E is *finite* if it is built up from the zero object by attaching a finite number of free cells.

Given a fibrant and cofibrant $G_\bullet$-spectrum E, we can consider the homotopy colimit

$$E^\flat := \operatorname*{hocolim}_{\alpha\in C_E} E_\alpha,$$

where the indexing category C_E is defined so that

- *objects* are maps $\alpha: E_\alpha \to W$ such that E_α is finite $G_\bullet$-spectrum, and
- *morphisms* are maps $f: E_\alpha \to E_\beta$ such that $\beta \circ f = \alpha$.

Lemma 4.1. *The category C_E is filtered.*

Proof. If $\alpha: E_\alpha \to E$ and $\beta: E_\beta \to E$ are objects, then the coproduct $\alpha+\beta: E_\alpha \vee E_\beta \to E$ is an object which is the base of a cone containing α and β.

Let $f, g: E_\alpha \to E_\beta$ be a pair of morphisms of C_E and let E_γ denote their equalizer in the category of $G_\bullet$-spectra. Then E_γ is finite and the base of a cone containing f and g. □

Lemma 4.2. *The evident map $E^\flat \to E$ is a weak equivalence.*

Proof. Let I_E denote the category whose objects are $\alpha: E_\alpha \to E$ such that α is an inclusion. Morphisms $E_\alpha \to E_\beta$ are also required to be inclusions.

Then the inclusion functor $I_E \to C_E$ is cofinal: (i) if $\alpha: E_\alpha \to E$ is an object of C_E, then the image of α is an object $\mathrm{im}\alpha \to E$ of I_E, and there is an evident morphism $E_\alpha \to \mathrm{im}\alpha$; (ii) if $t: B \to E$ is another object of I_E together equipped with a morphism $E_\alpha \to B$, then $\mathrm{im}\alpha \subset B$. These two properties show that $I_E \subset C_E$ is cofinal.

Consequently, we have an isomorphism of colimits (not homotopy colimits)

$$\operatorname*{colim}_{\alpha\in I_E} E_\alpha \cong \operatorname*{colim}_{\alpha\in C_E} E_\alpha.$$

But homotopy groups commute with filtered colimits, so one has that the colimit on the right is identified with the corresponding homotopy colimit.

On the other hand, the evident map

$$\operatorname*{colim}_{\alpha\in I_E} E_\alpha \to E,$$

is a weak equivalence because a map from a finite object into E automatically factors through this colimit. □

Remark 4.3. The previous lemma shows that we could have used I_E instead of C_E as the indexing category of the homotopy colimit. However, the reason why we use C_E is that the associated homotopy colimit is functorial.

4.3. A version for topological G-spectra

Suppose next that E is a (topological) G-spectrum, where $G = |G_\bullet|$. Let $E^{\mathrm{c,f}}$ be the result of making E functorially fibrant and cofibrant. Define sE to be $\operatorname{Sing} E^{\mathrm{c,f}} =$ the degreewise singular complex of $E^{\mathrm{c,f}}$. Then there is a functorial chain of weak equivalences from $|sE|$ to E. Moreover, sE is both fibrant and cofibrant.

Let $E \mapsto E^\sharp$ be the functor from G-spectra to G-spectra given by

$$E^\sharp := \operatorname*{hocolim}_{\alpha \in C_{sE}} |E_\alpha|.$$

Using Lemma 4.2, we have

Proposition 4.4. *The functor $E \mapsto E^\sharp$ admits a chain of natural weak equivalences to the identity.*

4.4. The functor $E \mapsto E^{t'G}$

Using the indexing convention in the colimit, define $E^{(\alpha)}$ to be the homotopy cofiber of the map $|\alpha| : |E_\alpha| \to |sE|$. Note that the homotopy colimit of the $E^{(\alpha)}$ is weakly contractible by Lemma 4.2 because there are cofibration sequences $|E_\alpha| \to |sE| \to E^{(\alpha)}$.

Let us define

$$E^{t'G} := \operatorname*{hocolim}_{\alpha} (E^{(\alpha)})^{hG},$$

where the homotopy colimit is indexed over the category C_{sE}.

Proposition 4.5. *There exists a natural weak equivalence*

$$E^{tG} \simeq E^{t'G},$$

where E^{tG} denotes the homotopy cofiber of the norm map.

Proof. For each index $\alpha \in C_{sE}$, we have a homotopy cofiber sequence

$$D_G \wedge_{hG} E^{(\alpha)} \to (E^{(\alpha)})^{hG} \to (E^{(\alpha)})^{tG}$$

Consequently, taking homotopy colimits, we obtain a homotopy cofiber sequence

$$\operatorname*{hocolim}_{\alpha} D_G \wedge_{hG} E^{(\alpha)} \to \operatorname*{hocolim}_{\alpha} (E^{(\alpha)})^{hG} \to \operatorname*{hocolim}_{\alpha} (E^{(\alpha)})^{tG}.$$

The first term is weakly contractible, since (i) the homotopy colimit of the $E^{(\alpha)}$ is weakly contractible and (ii) smash product with D_G and homotopy orbits commute with (homotopy) colimits. By definition, the second term of this sequence is $E^{t'G}$. We infer that the evident map

$$E^{t'G} \to \operatorname*{hocolim}_{\alpha} (E^{(\alpha)})^{tG}$$

is a weak equivalence.

Let $0 : * \to |sE|$ denote the map from the zero object. To complete the proof of the proposition, it will suffice to show that the inclusion

$$E^{tG} \simeq (E^{(0)})^{tG} \to \operatorname*{hocolim}_{\alpha}(E^{(\alpha)})^{tG}$$

is a weak equivalence. For this to be true, we need only show that the maps $(E^{(\alpha)})^{tG} \to (E^{(\beta)})^{tG}$ are weak equivalences for a morphism $f : E_\alpha \to E_\beta$. This follows from Axioms 1 and 4 by applying $(-)^{tG}$ to the homotopy cofiber sequence $C_f \to E^{(\alpha)} \to E^{(\beta)}$ and observing that the first term of the latter (= the homotopy cofiber f) is finite up to homotopy. □

Remark. In the above construction, if one replaces "finite" by "bounded" G-spectra, one obtains a possibly different theory $E \mapsto E^{fG}$ that satisfies axioms 1, 2' and 3. It can be shown (we omit the proof) that this variant coincides with Vogel cohomology [7] when G is discrete and E is an Eilenberg–MacLane spectrum.

5. Proof of Theorem A (uniqueness)

Let $E \mapsto E^{wG}$ be a functor satisfying Axioms 1–4. In particular, we have a natural map $E^{hG} \to E^{wG}$. Then, with respect to the notation of the previous section, we have maps

$$(E^{(\alpha)})^{hG} \to (E^{(\alpha)})^{wG} \xleftarrow{\simeq} (|sE|)^{wG} \simeq E^{wG},$$

where the second map is given by applying the functor $(-)^{wG}$ to the evident map $|sE| \to E^{(\alpha)}$ (the second displayed map is a weak equivalence by the same argument used in the proof of Proposition 4.5). To avoid notational clutter, in what follows we will abuse notation and identify E with $|sE|$. With respect to this change of notation, we have compatible maps $E \to E^{(\alpha)}$.

Since we have natural transformations $E^{hG} \to E^{tG}$ and $E^{hG} \to E^{wG}$, for each index α the diagram

$$\begin{array}{ccccc}
E^{tG} & \longleftarrow & E^{hG} & \longrightarrow & E^{wG} \\
\downarrow & & \downarrow & & \downarrow \\
(E^{(\alpha)})^{tG} & \leftarrow & (E^{(\alpha)})^{hG} & \rightarrow & (E^{(\alpha)})^{wG}
\end{array}$$

commutes. Consequently, there is a commutative diagram involving the associated homotopy colimits

$$\begin{array}{ccccc}
E^{tG} & \longleftarrow & E^{hG} & \longrightarrow & E^{wG} \\
\simeq\downarrow & & \downarrow & & \downarrow\simeq \\
\operatorname*{hocolim}_{\alpha}(E^{(\alpha)})^{tG} & \underset{\simeq}{\longleftarrow} & \operatorname*{hocolim}_{\alpha}(E^{(\alpha)})^{hG} & \rightarrow & \operatorname*{hocolim}_{\alpha}(E^{(\alpha)})^{wG}.
\end{array}$$

Set

$$E^{w'G} := \operatorname*{hocolim}_{\alpha}(E^{(\alpha)})^{wG}.$$

Then the diagram shows that we have a natural map

$$E^{t'G} \to E^{w'G}$$

which is compatible with the structure maps out of E^{hG}. Furthermore, the diagram shows that $E^{t'G}$ ($E^{w'G}$) is naturally weak equivalent to E^{tG} (resp. E^{wG}) by a chain of weak equivalences preserving the structure maps out of E^{hG}.

We now have a commutative diagram

$$\begin{array}{ccccc} E_{t'G} & \to & E^{hG} & \to & E^{t'G} \\ \downarrow & & \| & & \downarrow \\ E_{w'G} & \to & E^{hG} & \to & E^{w'G} \end{array}$$

in which the rows are homotopy fiber sequences, the left vertical map is the induced map of homotopy fibers, and the functors $E \mapsto E_{t'G}$ and $E \mapsto E_{w'G}$ commute with filtered homotopy colimits.

If E happens to be a finite G-spectrum, then the map $E_{t'G} \to E_{w'G}$ is a weak equivalence by the five lemma (since $E^{t'G}$ and $E^{w'G}$ are contractible in this instance, so the right vertical map of the diagram is a weak equivalence, and therefore the left one is too). Let E now be general. As $E \mapsto E_{t'G}$ and $E \mapsto E_{w'G}$ commute with filter homotopy colimits, and since E can be written up to weak equivalence as the filtered homotopy colimit of finite G-spectra, it follows that the map $E_{t'G} \to E_{w'G}$ is a weak equivalence for every E. Another application of the five lemma now shows that the map $E^{t'G} \to E^{w'G}$ is a weak equivalence for every E. This completes the proof of Theorem A. □

6. Proof of Theorem B

In this section we show that the norm map $D_G \wedge_{hG} \to E^{hG}$ is an assembly map for the homotopy fixed point functor $E \mapsto E^{hG}$. In Section 4 we defined a functor

$$E^{t'G} := \operatorname*{hocolim}_{\alpha \in C_{sE}} (E^{(\alpha)})^{hG},$$

where $E^{(\alpha)}$ denotes the homotopy cofiber of $|\alpha| : |E_\alpha| \to |sE|$. Moreover, we showed that the natural map $E^{hG} \mapsto E^{t'G}$ coincides with $E^{hG} \to E^{tG}$ in the homotopy category of functors (where E^{tG} as usual denotes the homotopy cofiber of the norm map).

Consequently, if we define

$$E_{t'G} := \operatorname*{hocolim}_{\alpha \in C_{sE}} (E_\alpha)^{hG},$$

it follows that the natural map

$$E_{t'G} \to E^{hG}$$

coincides with the norm map in the homotopy category of functors. It therefore suffices to show that the latter is an assembly map.

Let $G(E) \to E^{hG}$ be a natural transformation, where G is a homology theory. If we define G' to be the functor

$$G'(E) := \operatorname*{hocolim}_{\alpha \in C_{sE}} G(E_\alpha),$$

then the evident natural transformation $G'(E) \to G(E)$ is a weak equivalence. We also have an evident natural transformation $G'(E) \to E^{hG}$, and the diagram

$$\begin{array}{ccc} E_{t'G} & \to & E^{hG} \\ \uparrow & & \uparrow \\ G'(E) & \xrightarrow{\simeq} & G(E) \end{array}$$

is commutative. But this implies that the natural map $E_{t'G} \to E^{hG}$ is the universal left approximation to $E \mapsto E^{hG}$ by a homology theory in the homotopy category of functors. $\square$

7. Variants

The norm map $D_G \wedge_{hG} E \to E^{hG}$ generalizes the classically defined norm map $E_{hG} \to E^{hG}$ for finite groups G (it was shown in [8] that $D_G \simeq S^0$ when G is finite). Thus for infinite groups, one has to replace the homotopy orbit spectrum E_{hG} by a twisted version $D_G \wedge_{hG} E$ of it.

It is natural to ask whether it is possible to perform instead a modification of the codomain E^{hG} to obtain a norm map whose domain is E_{hG}. We answer this in the affirmative below, as a special case of a more general construction.

The idea of this more general construction is that one can precompose the homotopy fixed point functor with a homology theory from G-spectra to G-spectra. This yields another functor from G-spectra to spectra.

7.1. Twisted fixed points

Suppose that W denotes a $(G \times G)$-spectrum. In order to distinguish between the two actions of G, we write $G \times G$ as $G_\ell \times G_r$, where G_ℓ and G_r denote copies of G. Define the *twisted homotopy fixed point functor* $E \mapsto E^{h_W G}$ from G-spectra to spectra by the rule

$$E \mapsto (W \wedge_{hG_\ell} E)^{hG_r}.$$

Using the yoga of Sections 4–5, one obtains a norm (assembly) map

$$E_{\mathrm{t}_W G} \to E^{h_W G}$$

whose domain is the homotopy colimit

$$E_{\mathrm{t}_W G} := \operatorname*{hocolim}_{\alpha \in C_{sE}} (E_\alpha)^{h_W G}.$$

Example 7.1. We consider two special cases of the above. The first recovers the norm sequence of [8], whereas the second gives a norm map whose domain is the homotopy orbit construction.

(1) If $W = S^0[G]$ (the suspension spectrum of G_+), then $E^{h_W G} \simeq E^{hG}$ and the norm map in this case is just

$$D_G \wedge_{hG} E \to E^{hG}.$$

(2) Let $W = \hat{S}^0[G]$ denote the function spectrum $F(G_+, S^0)$ whose jth space is the space of stable maps $G_+ \to S^j$, with $G \times G$ acting via its action on the domain.

Note that $(S^0[G])^{h_W G} \simeq S^0$. More generally, one has $E^{h_W G} \simeq E_{hG}$ whenever E is G-finite. Using this identification, it is fairly straightforward to check that the domain of assembly map for the functor $E \mapsto E^{h_W G}$ is just $E \mapsto E_{hG}$, i.e., the homotopy orbit construction. The norm map in this example has the form

$$E_{hG} \to (\hat{S}^0[G] \wedge_{hG} E)^{hG}.$$

The composite of the latter with the forgetful map $(\hat{S}^0[G] \wedge_{hG} E)^{hG} \to \hat{S}^0[G] \wedge_{hG} E$ gives a *transfer map*

$$E_{hG} \to \hat{S}^0[G] \wedge_{hG} E.$$

If E is a finite G-spectrum and G is discrete, then the target of this transfer map can be interpreted as "locally finite homology of E with S^0-coefficients". This map is similar to a transfer map for infinite (discrete) groups defined by Carlsson [4] (the essential difference is that the target of Carlsson's transfer is locally finite homology with $\mathbb{Z}$-coefficients). Carlsson's transfer has been used to verify the K-theory Novikov conjecture for a wide class of groups [3].

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